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directly applicable) the settling down of fog makes a marked difference, more than is accounted for by the usual corrections given by Bessel.

A provisional table is appended. It must be taken as including seconds of arc, not because these are reliable, but because only thus could advantage be taken of the analysis of many observations.

Tentative Table

of correction to altitude due to shift of horizon caused by difference of temperature of sea and air (after applying to mean refraction the corrections for state of barometer and thermometer).

Sea colder, *plus* to alt.
Sea warmer, *minus* to alt.

Arguments :—Height of eye, feet
Difference of sea and air, degrees F.

Diff. Fahr.	Height of Eye, Feet.				Diff. Fahr.	Height of Eye, Feet.			
	0 or 140	20 or 120	40 or 100	60 or 80		0 or 140	20 or 120	40 or 100	60 or 80
°	′ ″	′ ″	′ ″	′ ″	°	′ ″	′ ″	′ ″	′ ″
1	11	25	31	35	7	1 20	1 35	1 43	1 50
2	23	36	43	48	8	1 31	1 47	1 55	2 2
3	34	48	55	1 0	9	1 43	2 0	2 7	2 15
4	46	1 0	1 7	1 13	10	1 54	2 12	2 19	2 27
5	57	1 11	1 19	1 25	20	3 48	4 6	4 19	4 31
6	1 8	1 23	1 31	1 37	30	5 42	6 3	6 19	6 35

Example :—Sea warmer by 23°, H.E. 30 feet

for 20° 4′ 12″
3 52
5 4 — to alt.

1906 *March* 15.

On Planetary Inversion. By F. J. M. Stratton, B.A.

I. *Introduction.*

§ 1. The question considered in the following paper is the influence of tidal friction in producing secular changes in the obliquities of the members of the solar system. This subject was suggested by some remarks of Professor W. H. Pickering in his account of the discovery of *Phœbe*, the ninth satellite of *Saturn.*[*] Discussing the retrograde direction of its motion in its orbit, he offered as an explanation the theory that the planets originally rotated on their axes in a retrograde direction, and that under the influence of tidal friction their axes of rotation had gradually tilted over into their present positions. If, then, a satellite were thrown off in a very early stage of the planet's evolution it would commence moving in a retrograde direction round the planet. If the oblateness of the planet were very small, or the satellite at a considerable distance from the planet's centre, the plane of the orbit of the satellite would not follow the plane of the planet's equator as it tilted over, but would fall back into a stable position near the ecliptic—a term used in this paper for the plane of the planet's orbit. Such a satellite would remain of the retrograde type exemplified by *Phœbe.* If, however, the satellite were evolved in a later stage of the planet's development (after the planet had greatly contracted and become more oblate), the satellite would move in an orbit whose stable position was almost coincident with the planet's equator, and the satellite would follow the planet's equator. Most of the known satellites of the solar system fall into this class.

Professor Pickering urged in support of this view that the classical nebular hypothesis, according to which the planets were thrown off in the form of rings, required an initial retrograde rotation of the planet and not a direct one, as Laplace assumed. But of recent years Sir George Darwin,[†] Professor T. C. Chamberlin,[‡] and Dr. F. R. Moulton[§] have adduced strong reasons for discarding the ring-theory, and it would seem that such confirmation as it would undoubtedly have given to this investigation must for the present be disregarded. Though apparently the classical form of the nebular hypothesis cannot now be accepted without considerable modifications I have here followed it in general as regards the history of the planetary sub-systems, and have assumed a planet to be a gradually contracting body, which from time to time may pass through a form of instability, resulting in the evolution of a satellite.

I have also assumed that the determining factor in producing secular changes in the obliquity of a planet has been tidal friction. A justification for this assumption may be found in a statement made by Adams[¶] that no secular alterations of a planet's obliquity can arise from the attraction of its satellites and the Sun. One reason for the further assumption of initial retrograde rotation is that it gives an explanation why the two outermost planets should have an obliquity greater than 90°. It

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[*] *Annals of Harvard College Observatory*, vol. liii. No. III., "The Ninth Satellite of *Saturn*," W. H. Pickering.

[†] Presidential Address to the British Association, South Africa, 1905, p. 17.

[‡] *Journal of Geology*, vol. viii.; and Carnegie Institution, Year Book, No. 3.

[§] *Astrophysical Journal*, 1900 February and 1905 October.

[¶] See p. 385.

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does not seem that under any theory of initial direct rotation their obliquities could have increased past 90° to the values that they appear to have at present. "It must be admitted," Sir George Darwin has said,* "that the Uranian system points to the possibility of the existence of a primitive planet with either retrograde rotation or at least with a very large obliquity of equator." At the same time I should feel considerable hesitation in applying to the inner planets a theory based on the position of the satellites of *Uranus* and *Neptune*, were it not that the remarkable discovery of *Phœbe* seems to compel us to attribute to *Saturn* also a primitive obliquity greater than 90°. And if once the possibility of such a change in the obliquity is conceded in *Saturn's* case the further extension of the theory to the inner planets presents no great difficulty. I must admit, however, that the whole theory is of an extremely speculative nature ; many serious difficulties still remain unsolved, such as the extension to heterogeneous planets of the theory of tides in a homogeneous, incompressible spheroid. If, therefore, I am betrayed in my paper into too dogmatic statements I hope it will be understood that they owe their emphatic form solely to a desire to avoid the continual use of the conditional mood. If I state as facts results indicated by the theory, it is not because the hazardous nature of the theory is not recognised, but because it becomes intolerable to continue to insist on the point throughout the whole paper. One word of defence should, perhaps, be added for the rather rough approximations employed from time to time. In treating a hypothesis which is certainly largely removed from the facts it does not seem necessary always to employ the methods of the most exact analysis. All that is required is to obtain the general conclusions to which the hypothesis leads.

§ 2. Leaving these general considerations on one side I proceed to discuss the paper in greater detail. Throughout I follow Sir George Darwin's theory of the action of the tides and I make free use of the results that he obtains in his series of papers in the *Philosophical Transactions* of the Royal Society.† The tides are regarded as a bodily deformation of the planet, which is supposed to be a homogeneous, incompressible, viscous spheroid.

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* "On the Secular Changes in the Elements of the Orbit of a Satellite revolving about a Tidally Distorted Planet," *Phil. Trans.* part ii. 1880, p. 886.

† "On the Bodily Tides of Viscous and Semi-elastic Spheroids and on the Ocean Tides upon a Yielding Nucleus," *ibid.* part i. 1879.

"On the Precession of a Viscous Spheroid and on the Remote History of the Earth," *ibid.* part ii. 1879.

"On Problems connected with the Tides of a Viscous Spheroid," *ibid.* part ii. 1879.

"On the Secular Changes in the Elements of the Orbit of a Satellite revolving about a Tidally Distorted Planet," *ibid.* part ii. 1880.

"On the Tidal Friction of a Planet attended by several Satellites and on the Evolution of the Solar System," *ibid.* part ii. 1882.

These papers will hereafter be referred to as "Tides," "Precession," "Problems," "Orbits," and "Evolution" respectively.

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Laplace has shown that, under certain conditions, compressibility does not affect the results to any great extent;* and Sir George Darwin has shown in "Evolution" that no serious error is involved in the hypothesis of homogeneity as opposed to heterogeneity, so long as only a surface layer of slight viscosity is affected. In the same paper he has shown that tidal friction can have but very slightly affected the dimensions of the planetary orbits. Hence I shall throughout consider the mean distances of the planets as having their present values. Further, I adopt Sir George Darwin's limitation in considering the eccentricities zero. In adapting his results to my hypothesis I shall of course be unable to reproduce all his reasoning, and I shall have to ask the reader, if unacquainted with his papers, to take my quotations from them as established.

In fitting this theory on to Sir George Darwin's investigations as to the tidal evolution of the Earth-Moon system I have to show that, starting with a planet unattended by a satellite, rotating in a retrograde direction and revolving about the Sun, it is possible to proceed to a state of affairs in which the planet in a highly viscous state rotates some 3,000 times in the course of one revolution about the Sun, and has an obliquity of about 11°. Part II. is therefore devoted to a discussion of the secular changes in the obliquity of a viscous spheroid due to the tidal action of one satellite (the Sun). This is practically the problem dealt with in "Precession," but it is here extended to the case where the obliquity is greater than 90°.

In Part III. I introduce a second satellite, small in mass compared with the planet and much nearer than the Sun, and I discuss the influence of the planet's tides on the inclination of the satellite and on the obliquity of the planet. In Part IV. I extend the last case to admit of several small satellites being present, and I make an application of the theory to the satellites of the solar system. In this part I extend the practical applications of the theory by a discussion of the past changes suggested by the theory in the obliquities of the planets and in the orbits of the satellites. Finally in Part V. I give a brief summary of the results obtained in the foregoing paper and of the general conclusions to which the theory points.

§ 3. I must now explain the notation, which, following Sir George Darwin, I shall use throughout the paper. The method employed is that of the disturbing function, changes in the elements fixing a body's position being obtained by differentiating the function with respect to certain of the other elements expressing the position of the disturbed body. Now we have to determine the perturbations of a satellite under the influence of tides raised by itself, and thus the elements of the satellite's path will enter the disturbing function in two ways, in the expression both of the distorted state of the planet and of the position of the satellite. It is only in this latter sense that we require the function to be differentiated, and hence we make the following convention : the elements of the tide-raising satellite are to be unaccented letters, those of the disturbed satellite (or, if the effect of the planet is being considered, of the disturbing satellite) are to be accented ; the differentiation is to be made with regard to the accented letters, and after the differentiation the two satellites may, if necessary, be made identical by dropping the accents, or they may be kept distinct.

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* *Mécanique Céleste*, livre iv. ch. v.

The following are the symbols employed :—

$t =$ the time.

The suffix $_0 =$ the initial value of the symbol to which it is attached.

For the planet we have :

$M =$ mass.

$a =$ mean radius.

$e =$ ellipticity of a meridian section.

$w =$ density of the planet supposed homogeneous.

$g' =$ mean gravity at its surface.

$g = \dfrac{2}{5}\dfrac{g'}{a}$.

$n =$ angular velocity of diurnal rotation.

$\psi =$ longitude of vernal equinox measured along the ecliptic —some fixed plane regarded as practically coincident with the planet's orbit—from a fixed point in the ecliptic.

$i =$ obliquity of ecliptic.

$\chi =$ angle between a point fixed on the equator and the vernal equinox.

For the tide-raising satellite we have :

$c =$ mean distance.

$\xi = \left(\dfrac{c}{c_0}\right)^{\frac{1}{2}}$.

$\Omega =$ mean motion.

$j =$ inclination of orbit to ecliptic.

$N =$ longitude of node.

$\epsilon =$ longitude of epoch.

$m =$ mass.

$\nu =$ ratio of planet's mass to satellite's mass $= \dfrac{M}{m}$.

$\tau = \dfrac{3\Omega^2}{2(1+\nu)}$.

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For the disturbed (or disturbing) satellite the same symbols apply, but accented.

Further, we have a quantity k such that $\frac{kn}{\xi}$ = the ratio of the moment of momentum of the planet's rotation to that of the orbital motion of planet and satellites round their common centre of inertia. If W is the disturbing function the equation giving the variation of the inclination of the satellite's orbit is

$$-\frac{dj}{dt}=\frac{k}{\xi}\left(\frac{1}{\sin j'}\frac{\partial W}{\partial N'}+\tan\tfrac{1}{2}j'\frac{\partial W}{\partial \epsilon'}\right) \quad \ldots \quad \ldots \quad (1)$$

The equation for the variation of the obliquity is

$$n\sin i\frac{di}{dt}=\cos i\frac{\partial W}{\partial \chi'}-\frac{\partial W}{\partial \psi'} \quad \ldots \quad \ldots \quad (2)$$

These two equations are quoted from "Orbits."

Before giving Professor Darwin's value for W several fresh definitions must be given. There are present in the disturbing function tides of seven distinct speeds* ($2n-2\Omega$, $2n$, $2n+2\Omega$ $n-2\Omega$, n, $n+2\Omega$, 2Ω), and the tides are retarded by different amounts and are of varying heights. The retardation of the phase of the tide is called its "lag," and its height is expressed as a fraction of the corresponding equilibrium tide in a perfectly fluid spheroid. The following table gives the symbols to be employed for the several tides :—

Tide.	Semi-diurnal.			Diurnal.			Fortnightly.
	Slow.	Sidereal.	Fast.	Slow.	Sidereal.	Fast.	
Speed	$2n-2\Omega$	$2n$	$2n+2\Omega$	$n-2\Omega$	n	$n+2\Omega$	2Ω
Height	F_1	F	F_2	G_1	G	G_2	H
Lag	$2f_1$	$2f$	$2f_2$	g_1	g †	g_2	$2h$

The heights and lags are connected by the following relations :—

$$\left.\begin{array}{r} F_1=\cos 2f_1\,;\ F=\cos 2f\,;\ F_2=\cos 2f_2\,;\ G_1=\cos g_1\,; \\ G=\cos g\,;\ G_2=\cos g_2\,;\ H=\cos 2h \\ \tan 2f_1=2(n-\Omega)/\sigma\,;\tan 2f=2n/\sigma\,;\tan 2f_2=2(n+\Omega)/\sigma\,; \\ \tan g_1=(n-2\Omega)/\sigma\,;\ \tan g=n/\sigma\,;\ \tan g_2=(n+2\Omega)/\sigma\,; \\ \tan 2h=2\Omega/\sigma \end{array}\right\}(3)$$

where $\sigma=2g'\,a\,\omega/19v$.

v is the coefficient of viscosity of the tidally disturbed layer, and hence we may take f roughly as a measure of the viscosity of the planet ; $f = 10°$ represents a small viscosity, $f = 45°$ an infinite one.

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* The "speed" of a tide is the coefficient v of t in the trigonometrical term $\cos(vt+p)$ occurring in the expression for the tide's action.

† Note that this is not the same g as the one defined above, p. 378. As this g is an angle and always appears in a trigonometrical form there is no chance of confusion between the two g's.

$$\left.\begin{aligned} &\text{Let now } \mathrm{P} = \cos\frac{i}{2},\ \mathrm{Q} = \sin\frac{i}{2},\ p = \cos\frac{j}{2},\ q = \sin\frac{j}{2} \\ &\varpi = \mathrm{P}p - \mathrm{Q}q\, e^{\sqrt{-1}(\mathrm{N}-\psi)} = \mathrm{P}p - \mathrm{Q}q \operatorname{expi}\{\mathrm{N}-\psi\}\ ^* \\ &\qquad\qquad k = \mathrm{Q}p + \mathrm{P}q \operatorname{expi}\{\mathrm{N}-\psi\} \\ &\underline{\varpi} = \mathrm{P}p - \mathrm{Q}q \operatorname{expi}\{-(\mathrm{N}-\psi)\}\ \ \underline{k} = \mathrm{Q}p + \mathrm{P}q \operatorname{expi} \\ &\qquad\qquad\qquad \{-(\mathrm{N}-\psi)\} \end{aligned}\right\} \quad . \quad (4)$$

Then $\mathrm{W} = (\mathrm{W}_{\mathrm{I}} + \mathrm{W}_{\mathrm{II}} + \mathrm{W}_{\mathrm{III}} + \mathrm{W}_1 + \mathrm{W}_2 + \mathrm{W}_3 + \mathrm{W}_0)$ consists of the seven following pairs of terms :—

$\frac{g}{\tau\tau'}\mathrm{W}_{\mathrm{I}}$ = slow semi-diurnal term

$$= \tfrac{1}{4}\mathrm{F}_1\varpi^4\underline{\varpi}'^4 \operatorname{expi} 2\{\chi - \chi' - (\epsilon - \epsilon') + \psi - \psi' - f_1\} + \dots$$

$\frac{g}{\tau\tau'}\mathrm{W}_{\mathrm{II}}$ = sidereal semi-diurnal term

$$= \mathrm{F}\varpi^2\underline{k}^2\underline{\varpi}'^2k'^2 \operatorname{expi} 2(\chi - \chi' - f) + \dots$$

$\frac{g}{\tau\tau'}\mathrm{W}_{\mathrm{III}}$ = fast semi-diurnal term

$$= \tfrac{1}{4}\mathrm{F}_2\underline{k}^4k'^4 \operatorname{expi} 2\ \{\chi - \chi' + (\epsilon - \epsilon') - (\psi - \psi') - f_2\} + \dots$$

$\frac{g}{\tau\tau'}\mathrm{W}_1$ = slow diurnal term

$$= \mathrm{G}_1\varpi^3k\underline{\varpi}'^3\underline{k}' \operatorname{expi}\ \{\chi - \chi' - 2(\epsilon - \epsilon') + 2(\psi - \psi') - g_1\} + \dots$$

$\frac{g}{\tau\tau'}\mathrm{W}_2$ = sidereal diurnal term

$$= \mathrm{G}(\varpi\underline{\varpi} - k\underline{k}(\varpi'\underline{\varpi}' - k'\underline{k}')[\varpi\underline{k\varpi}'k' \operatorname{expi}\ \{\chi - \chi' - g\} + \dots]$$

$\frac{g}{\tau\tau'}\mathrm{W}_3$ = fast diurnal term

$$= \mathrm{G}_2(\underline{\varpi k}^3\varpi'k'^3 \operatorname{expi}\{\chi - \chi' + 2(\epsilon - \epsilon') - 2(\psi - \psi') - g_2\} + \dots$$

$\frac{g}{\tau\tau'}\mathrm{W}_0$ = fortnightly term

$$\begin{aligned} &= \tfrac{3}{2}(\tfrac{1}{3} - 2\varpi\underline{\varpi}k\underline{k})(\tfrac{1}{3} - 2\varpi'\underline{\varpi}'k'\underline{k}') \\ &\qquad + \tfrac{3}{2}\mathrm{H}\underline{\varpi}^2\underline{k}^2\varpi'^2k'^2 \operatorname{expi} 2\{\epsilon - \epsilon' - (\psi - \psi') - h\} + \dots \end{aligned}$$

The second term in each of the above expressions can be obtained from the first by exchanging ϖ with ϖ', $\underline{\varpi}$ with $\underline{\varpi}'$, k with k', and $\underline{k}$ with $\underline{k}'$, and also changing the sign of the exponential expression. This gives the disturbing function to be differentiated, as

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* For convenience of printing we write expi for $e^{\sqrt{-1}}$. This abbreviation has been suggested by Professor Turner.

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in equations (1) and (2), for the secular variations in the elements of the satellite or the planet.

II. *Secular Changes in the Obliquity of a Viscous Spheroid under the Tidal Action of One Satellite.*

§ 1. This problem has already been discussed by Sir George Darwin in "Precession," and we shall make use of the result he there obtains (p. 474) for the secular rate of variation of the obliquity.

We shall have then

$$\left.\begin{aligned}\frac{di}{dt}=\frac{\tau^2}{gn}\frac{1}{128}\sin i\{&[10\sin 4f_1-10\sin 4f_2+16\sin 2g_1-16\sin 2g_2-12\sin 4h]\\&+\cos i[15\sin 4f_1-4\sin 4f+15\sin 4f_2+18\sin 2g_1-24\sin 2g+18\sin 2g_2]\\&+\cos 2i[6\sin 4f_1-6\sin 4f_2+12\sin 4h]\\&+\cos 3i[\sin 4f_1+4\sin 4f+\sin 4f_2-2\sin 2g_1-8\sin 2g-2\sin 2g_2]\},\end{aligned}\right\}\quad(5)$$

where f, f_1, f_2, g, g_1, g_2, h are determined by equation (3) when some value is assumed for one of them. The extension of Sir George Darwin's work in this paper consists in two things. He deals with values of i from 0° to 90° only, while here i must be taken to vary between 0° and 180°. Again, he has not to consider the question of the variation in the viscosity, or rather in the angle $2f$, as the planet contracts from a nebulous mass and increases its rate of rotation. This point will first be briefly discussed.

If we consider the changes in the viscosity that accompany the evolution of the planet, we find that sufficient data are lacking to enable us to formulate any law on the subject. As will be seen later, the only planets in which the Sun's action on its own tides has been the determining factor throughout in producing changes of obliquity are *Mercury*, *Venus*, the Earth (before the birth of the Moon), and *Mars*. The obliquity of *Mercury* is unknown, that of *Venus* doubtful, different observers giving values as different as 37° and 0°; the obliquity of the Earth when revolving in a highly viscous state at the time of the Moon's birth was about 11°; the present obliquity of *Mars* is nearly 25°. A glance at the details given below will show that, starting with a retrograde rotation, tidal friction would eventually drive the axis of rotation over into a stable position where the obliquity lies somewhere between 90° and 0°. For all but viscosities so large that the semi-annual tide is of greater importance than the diurnal and semi-diurnal tides the stable value of the obliquity will be near 90°. The low obliquities of the Earth and *Mars* can only be obtained if we make the assumption that some such viscosities have obtained in their past history; in picking out a few cases for arithmetical investigation I have been guided rather by this idea than by any supposed law

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connecting viscosity with density. The value of the results given lies chiefly in the illustration they afford of the general nature of the graphs for $\frac{di}{dt}$.

§ 2. The accompanying figure gives the form of the graph for $\frac{di}{dt}$ in the following cases :—

$n = 5\Omega$	$h = 10^\circ$	$f = 30^\circ\ 36'$
$n = 10$	$h = 11$	$f = 38\ \ 3$
$n = 20$	$h = 12\ \ 30$	$f = 41\ \ 58$
$n = 100$	$h = 15$	$f = 44\ \ 30\frac{1}{4}$
$n = 1{,}000$	$h = 20$	$f = 44\ \ 58$
$n = 10{,}000$	$h = 22\ \ 30$	$f = 44\ \ 59\frac{3}{4}$

From the figure we see fairly satisfactorily that for varying values of $\frac{n}{\Omega}$ and for different viscosities we may make $\frac{di}{dt}$ zero for any value of i we like between 0° and 90°; also, since in passing through this zero value $\frac{di}{dt}$ becomes negative as i increases, the obliquity will there have a stable value, but not at 0° nor at 180°. Starting, then, with a sphere of slight viscosity, with an obliquity of 180°, we shall have initially a decrease of obliquity, and in any case the obliquity will eventually become less than 90°. But as the sphere contracts its viscosity increases, and if it does so in such a way that the effect of the semi-annual tide steadily increases relatively to that of the diurnal and semi-diurnal tides, then, when $\frac{n}{\Omega}$ has attained a large value, the sphere will rotate as though almost rigid for these latter tides and the stable position of the obliquity will be a very small angle. If, however, after the planet reaches this position the viscosity still increases for the bodily tides, but an ocean of much smaller viscosity forms upon the surface, then the effect of the bodily tides may become negligible, and under the influence of the slightly viscous ocean tides the planet may gradually tilt over till it reaches a position of stability with a fairly high obliquity.

Thus we may conceive that *Mercury* has tilted over into a position of small obliquity and that it never had oceans of sufficient magnitude to tilt it back. *Venus* may or may not have been seriously affected by ocean tides after reaching a position of low obliquity under the influence of bodily tides. Here observation leaves it open for one to hold either view. The Earth has in its later history been affected by the presence of the Moon, and therefore it need not be considered in this section. *Mars* may never have tilted over so far as the Earth did, owing to its

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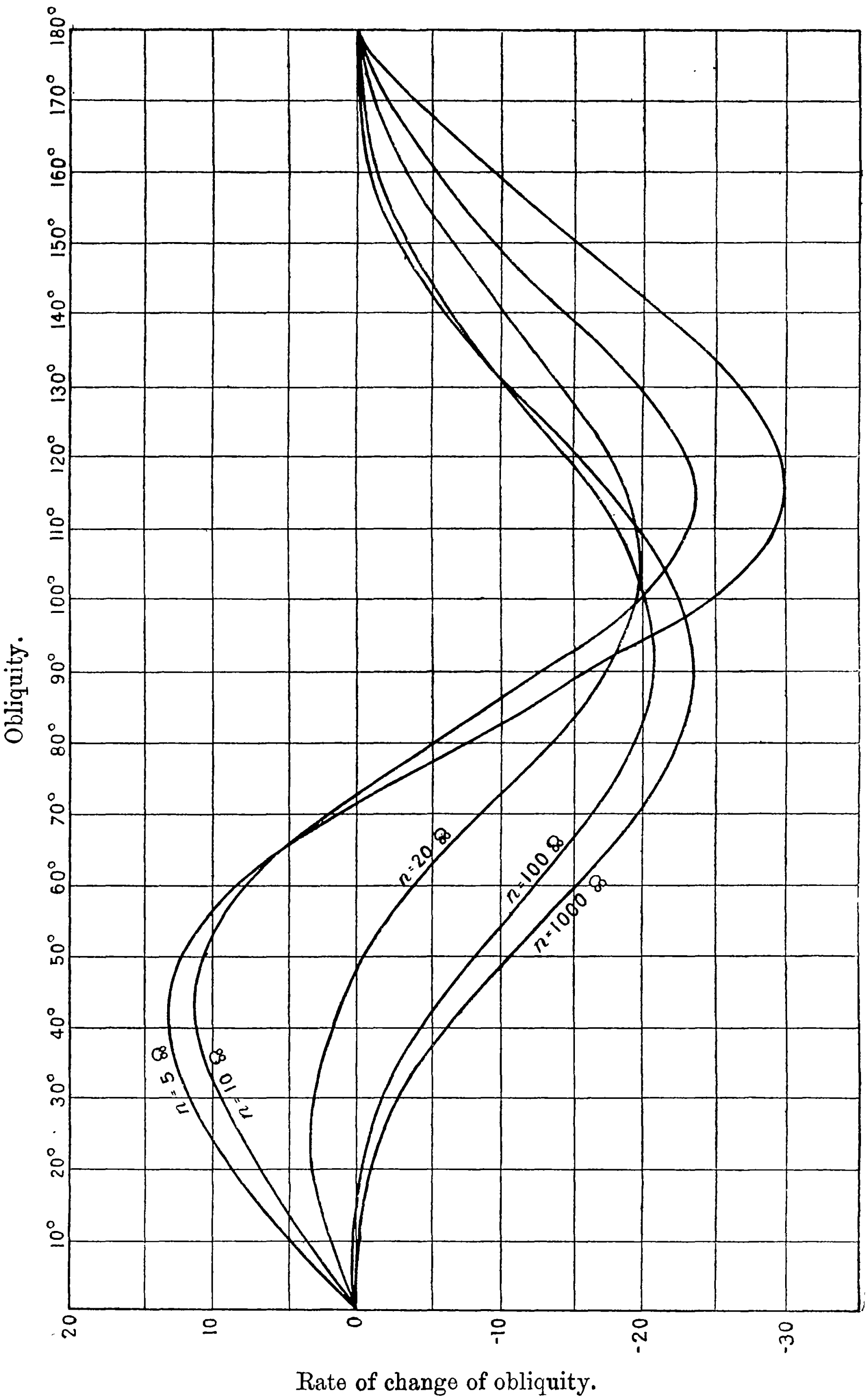

Diagram showing change of obliquity of a planet due to solar tides.

longer period of revolution, and it may be already on the way back, under the influence of ocean tides, to a position of stability with its axis of rotation almost in the plane of its orbit. The other planets will be considered in Part IV.

III. *The Tidal Action of Two Satellites.*

§ 1. When there are two satellites, one of which we distinguish as the Sun, setting up tides on a planet and acting on their own and each other's tides, we have to distinguish four sets of terms in the disturbing function which gives the perturbations of the planet and the satellite :—

(i) when the tide-raiser is the satellite and the disturber is the satellite,
(ii) when the tide-raiser is the Sun and the disturber is the satellite,
(iii) when the tide-raiser is the satellite and the disturber is the Sun,
(iv) when the tide-raiser is the Sun and the disturber is the Sun.

For perturbations in the planet all four sets of terms exist, but for perturbations in the satellite we alter "disturber" into "disturbed body," and notice that we have only to consider cases (i) and (ii).

Now let τ as defined on p. 378 refer to the satellite and let τ' refer to the Sun. Then, in considering the changes in the satellite's inclination given by equation (1), we derive from the disturbing function two sets of terms corresponding to cases (i) and (ii) above. These consist of functions of i and j of somewhat the same type as that given in equation (5).* But they have as coefficients the factors $\frac{k}{\xi}\frac{\tau^2}{g}$ and $\frac{k}{\xi}\frac{\tau\tau_s}{g}$ respectively, and these factors are so small for all the satellites of the solar system as to render these terms in the disturbing function negligible compared with certain other terms yet to be mentioned.

For the disturbing function (†)

$$\Omega = \mathrm{K}\cos^2\gamma + \mathrm{K}'\cos^2\gamma'$$

where γ, γ' are the angles made by the satellite's orbit with the planet's orbit and equator respectively, from which, according to the ordinary dynamical theory, the secular position of the satellite's orbit (*i.e.* the satellite's proper plane, round the pole of which the pole of the orbit describes an ellipse) is derived, gives

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* The exact form of the expression derived from case (i) is given in "Orbits," pp. 740, 746.

† See Tisserand, *Mécanique Céleste*, vol. iv. ch. vi. s. 37.

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some secular terms in the expression for $\frac{dj}{dt}$. These give the following equation for the position of the satellite's proper plane :

$$\mathrm{K} \sin 2j = \mathrm{K}' \sin 2(i-j),$$

or, as it may be written,

$$\tan 2(i-j) = \frac{\mathrm{K} \sin 2i}{\mathrm{K}' + \mathrm{K} \cos 2i} \quad \text{or} \quad \tan 2j = \frac{\mathrm{K}' \sin 2i}{\mathrm{K} + \mathrm{K}' \cos 2i}.$$

The addition to the disturbing function of the tidal terms mentioned in the preceding paragraph will not sensibly affect the equilibrium position of the satellite's orbit. For if $(i-j)$ is small compared with i—i.e. if $\frac{\mathrm{K}}{\mathrm{K}'}$ is small—then the numerator of the above expression for tan $2\,(i-j)$ should be diminished by a quantity of the order of $\left(\frac{a}{c}\right)^5 \mathrm{K} \sin 2i$, and the denominator increased by a quantity of the order of $\frac{a^3}{ec_s{}^3}\,\mathrm{K}'$, where a is the planet's mean radius, c, c_s the satellite's and the Sun's mean distances, and e is the planet's oblateness. These alterations will produce a quite insensible diminution in the angle $(i-j)$; in the other case, where $\frac{\mathrm{K}'}{\mathrm{K}}$ is near unity or small, the additional terms due to tidal action are still less sensible than in the above case. In all cases, then, tidal action may be neglected in finding the secular position of a satellite's orbit, which is found by the usual gravitational methods; it is different, however, when we consider the obliquity of a planet.

§ 2. Adams has shown* that the obliquity of a planet is only altered by periodic terms under the attraction of its satellites and the Sun, so that we must look to other causes for secular changes in the obliquity. This being the case it seems fair to assume that the determining factor in the obliquity of a planet is tidal friction, and that is, in fact, the fundamental assumption on which this whole paper is based. For perturbations in the planet we have all the four sets of terms arising from the cases discussed on p. 384 ; as they have respectively the coefficients $\frac{\tau^2}{g}$, $\frac{\tau\tau_s}{g}$, $\frac{\tau\tau_s}{g}$, $\frac{\tau_s{}^2}{g}$ we see that we need take into consideration the tidal effect of the satellite, only when $\frac{\tau}{\tau_s}$ is large or near unity. When $\frac{\tau}{\tau_s}$ is small the planet moves as though under solar tidal friction alone, in the manner discussed in Part II. Now, except for the Moon, in all cases where $\frac{\tau}{\tau_s}$ is finite or large we have $\frac{\mathrm{K}'}{\mathrm{K}}$ also

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* *Observatory*, 1886 April.

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large, and the satellite's orbit has a secular position which passes through the node of the planet's equator on the planet's orbit, and is inclined at a very small angle to the planet's equator. That is to say, that in all cases where we have to take into consideration the presence of a second satellite in discussing the change in a planet's obliquity we may take $(i-j)$ small compared with i, and $N-\psi = 180°$,* for N, ψ are the longitudes on the planet's orbit of the vernal equinox and the ascending node of the satellite's orbit.

In this case $\quad \operatorname{expi}(N-\psi) = \operatorname{expi}\{-(N-\psi)\} = -1$;

therefore for the satellite

$$\varpi = Pp - Qq \operatorname{expi}(N-\psi) = \cos\frac{i}{2}\cos\frac{j}{2} + \sin\frac{i}{2}\sin\frac{j}{2}$$
$$= \cos\tfrac{1}{2}(i-j) = \underline{\varpi}$$
$$k = Qp + Pq \operatorname{expi}(N-\psi) = \sin\frac{i}{2}\cos\frac{j}{2} - \cos\frac{i}{2}\sin\frac{j}{2}$$
$$= \sin\tfrac{1}{2}(i-j) = \underline{k},$$

and therefore, since $(i-j)$ is small, k is small, and it is easier in discussing $\frac{di}{dt}$ to keep k and ϖ in our formulæ to the very last and then to substitute. Taking then the various terms of W from p. 380, and substituting in the equation (2), which we wrote in the form

$$n\frac{di}{dt} = \frac{P^2-Q^2}{2PQ}\frac{\partial W}{\partial \chi'} - \frac{1}{2PQ}\frac{\partial W}{\partial \psi'},$$

we derive sets of terms from the four cases as follows :—

Case (i).—Since ψ' occurs only with N' in the ϖ's and k's, and with ϵ' in the other exponential factors, and since N', ϵ' only occur in the connection $N'-\psi'$, $\epsilon'-\psi'$ it really facilitates the finding of the result to write :

$$\frac{\partial W}{\partial \psi'} = -\left(\frac{\partial W}{\partial N'} + \frac{\partial W}{\partial \epsilon'}\right)$$

For the sake of brevity in the following work we shall drop the factor $\frac{\tau^2}{g}$ and let ϵ, N, ϵ', N' stand for $\epsilon-\psi$, $N-\psi$, $\epsilon'-\psi'$, $N'-\psi'$ respectively, and also omit the second terms in W_1, &c. After differentiation we can drop the accents, as the tide-raiser and disturber are the same body. In this case k', $\underline{k}'$ have the same value as k, $\underline{k}$ or $\sin\frac{1}{2}(i-j)$, while ϖ', $\underline{\varpi}'$ have the same value as ϖ, $\underline{\varpi}$ or $\cos\frac{1}{2}(i-j)$. Since we are only seeking the first approximation we shall only discuss the terms in W, which can give us terms of the lowest possible order, namely, the first, in

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* See p. 389.

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k, k' ; the next lowest order occurring is the third, and terms of that order may safely be neglected.

Now

$$W_{\mathrm{I}} = \tfrac{1}{4}F_{\mathrm{I}}(\varpi^4\underline{\varpi}'^4 \operatorname{expi} 2\{\chi-\chi'-(\epsilon-\epsilon')-f_{\mathrm{I}}\} + \ldots$$

$$\therefore \frac{\partial W_{\mathrm{I}}}{\partial N'} = \sqrt{-1}\,qQF_{\mathrm{I}}(-\varpi^7 \operatorname{expi}(-2f_{\mathrm{I}})+\varpi^7 \operatorname{expi} 2f_{\mathrm{I}})$$

$$\frac{\partial W_{\mathrm{I}}}{\partial \epsilon'} = \frac{\sqrt{-1}}{1} F_{\mathrm{I}}(\varpi^8 \operatorname{expi}(-2f_{\mathrm{I}})-\varpi^8 \operatorname{expi} 2f_{\mathrm{I}})$$

$$\frac{\partial W_{\mathrm{I}}}{\partial \chi'} = -\frac{\partial W_{\mathrm{I}}}{\partial \epsilon'}$$

$$n\frac{di_{\mathrm{I}}}{dt} = \frac{1}{2PQ}\left\{2Q^2\frac{\partial W_{\mathrm{I}}}{\partial \epsilon'}+\frac{\partial W_{\mathrm{I}}}{\partial N'}\right\}$$

$$= -F_{\mathrm{I}} \sin 2f_{\mathrm{I}}\varpi^7\left(\frac{q}{P}-\frac{Q}{P}\varpi\right)$$

$$= F_{\mathrm{I}} \sin 2f_{\mathrm{I}}\varpi^7 k.$$

W_{II} must give us as the largest possible term one in k^3 ; it is therefore to be neglected here, as also is W_{III}, which gives terms of the seventh order in k.

$$W_{\mathrm{I}} = G_{\mathrm{I}}(\varpi^3 k\underline{\varpi}'^3\underline{k}' \operatorname{expi}\{\chi-\chi'-2(\epsilon-\epsilon')-g_{\mathrm{I}}\} + \ldots$$

$$\therefore \frac{\partial W_{\mathrm{I}}}{\partial N'} = 2G_{\mathrm{I}} \sin g_{\mathrm{I}}\varpi^5 k(-3kqQ+\varpi qP)$$

$$\frac{\partial W_{\mathrm{I}}}{\partial \epsilon'} = 4G_{\mathrm{I}} \sin g_{\mathrm{I}}\varpi^6 k^2$$

$$\frac{\partial W_{\mathrm{I}}}{\partial \chi'} = -\frac{1}{2}\frac{\partial W_{\mathrm{I}}}{\partial \epsilon'}$$

and

$$n\frac{di_{\mathrm{I}}}{dt} = \frac{1}{2PQ}\left\{\frac{\partial W_{\mathrm{I}}}{\partial N'}+\frac{1}{2}\frac{\partial W_{\mathrm{I}}}{\partial \epsilon'}(3Q^2+P^2)\right\}$$

$$= -G_{\mathrm{I}} \sin g_{\mathrm{I}}\varpi^5 k\left\{\frac{3k}{P}(q-Q\varpi)-\frac{\varpi}{Q}(q+Pk)\right\}$$

$$= G_{\mathrm{I}} \sin g_{\mathrm{I}}\varpi^5 k(3k^2+\varpi^2)$$

$$W_2 = G(\varpi\underline{\varpi}-k\underline{k})(\varpi'\underline{\varpi}'-k'\underline{k}')(\varpi\underline{k}\,\underline{\varpi}'k' \operatorname{expi}\{\chi-\chi'-g\} + \ldots$$

$$\frac{\partial W_2}{\partial N'} = 2G \sin g\,\varpi k(\varpi^2-k^2)^2(-kqQ-\varpi qP)$$

$$\frac{\partial W_2}{\partial \epsilon'} = 0$$

$$\frac{\partial W_2}{\partial \chi'} = -2G \sin g\varpi^2 k^2(\varpi^2-k^2)^2$$

$$\therefore n\frac{di_2}{dt} = -G \sin g\varpi k(\varpi^2-k^2)^2\{pq+\varpi k(P^2-Q^2)\}\frac{1}{PQ}$$

$$= -G \sin g\varpi k(\varpi^2-k^2)^3.$$

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W_3 may be neglected for the same reason as W_{III}, and so too may W_o, for the terms of lowest order derived from it are of the third order in k.

Hence, retaining only the terms of the lowest order in k and restoring the omitted factor, we obtain from this case :—

$$n\frac{di_{(i)}}{dt} = \frac{\tau^2}{g}k\varpi^7(F_1 \sin 2f_1 + G_1 \sin g_1 - G \sin g)$$

$$= \frac{\tau^2}{g}\sin \tfrac{1}{2}(i-j)(F_1 \sin 2f_1 + G_1 \sin g_1 - G \sin g).$$

Case (ii).—In this case only W_{II}, W_2, W_o can give secular terms in $\frac{di}{dt}$: here ϖ', k', $\underline{\varpi}'$, $\underline{k}'$ have the same value as in case (i), while

$$\varpi = \underline{\varpi} = \cos \tfrac{1}{2}i$$

$$k = \underline{k} = \sin \tfrac{1}{2}i.$$

The term of highest order of magnitude is the term independent of k' arising from W_2, and that is the only term we shall consider here. It gives :—

$$n\frac{di_{(ii)}}{dt} = -\frac{\tau\tau_s}{g}G \sin g\varpi k(\varpi^2 - k^2)(\varpi'^2 - k'^2)\frac{pq}{PQ}$$

$$= -\frac{\tau\tau_s}{g}G \sin g \cos\frac{i}{2}\sin\frac{i}{2}\cos i$$

$$= -\tfrac{1}{4}\frac{\tau\tau_s}{g}G \sin g \sin 2i.$$

Case (iii).—Here we have

$$\varpi = \underline{\varpi} = \cos \tfrac{1}{2}(i-j) \qquad \varpi' = \underline{\varpi}' = \cos \tfrac{1}{2}i$$

$$k = \underline{k} = \sin \tfrac{1}{2}(i-j) \qquad k' = \underline{k}' = \sin \tfrac{1}{2}i.$$

Hence, since the coefficient is the same $\left(\frac{\tau\tau_s}{g}\right)$ as in the previous case we need only consider terms independent of k. Since no such term arises from either W_{II}, W_2, or W_o, the only three parts of W which can give secular terms in this case, $\frac{di_{(iii)}}{dt}$, is negligible compared with $\frac{di_{(ii)}}{dt}$.

Case (iv) is the case dealt with in Part II., and the result obtained is given in equation (5).

We may greatly simplify the final expression for $\frac{di}{dt}$ by writing $f = f_1 = f_2$, $g = g_1 = g_2$ throughout. For in the only

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case where $\frac{n}{\Omega}$ is moderately small, so that the approximation might make a sensible difference, the factor $\sin(i-j)$ is so small as to render the term altogether negligible.

We obtain then for $\frac{di}{dt}$ in the presence of two satellites, one of which is the Sun, the equation

$$\left.\begin{aligned} 2gn\frac{di}{dt} &= \tau^2 \sin \tfrac{1}{2}(i-j) \sin 4f - \tfrac{1}{4}\tau\tau_s \sin 2i \sin 2g \\ &\quad + \frac{\tau_s^2}{64} \sin i \{-12 \sin 4h + \cos i(26 \sin 4f + 12 \sin 2g) \\ &\qquad + \cos 2i \,.\, 12 \sin 4h + \cos 3i(6 \sin 4f - 12 \sin 2g)\} \end{aligned}\right\} \cdots \quad (6)$$

There is one correction to the above result which seems well worth noting. We have tacitly assumed that, because $(\mathrm{N}-\psi)$ oscillates through a value 180°, we may take 0 and -1 as the mean values of $\sin(\mathrm{N}-\psi)$ and $\cos(\mathrm{N}-\psi)$ respectively. This is of course not exact for $\cos(\mathrm{N}-\psi)$, since its deviations from -1 are always of the same sign, and we must take as its mean value $(-1+\eta)$. In this case we must write

$$\varpi = \cos \tfrac{1}{2}(i-j) - \eta \sin \frac{\dot{i}}{2} \sin \frac{\dot{j}}{2} = \underline{\varpi}$$

$$k = \sin \tfrac{1}{2}(i-j) + \eta \cos \frac{\dot{i}}{2} \sin \frac{\dot{j}}{2} = \underline{k}.$$

Then retaining the terms in η of the first order from ϖ, $\underline{\varpi}$, k, $\underline{k}$ we have the following corrections to make: In case (i) the factor $\sin \frac{1}{2}(i-j)$ in the expression for $n\frac{di_{(1)}}{dt}$ is to be replaced by $\left(\sin \frac{1}{2}(i-j) + \eta \cos \frac{\dot{i}}{2} \sin \frac{\dot{j}}{2}\right)$. In case (ii) . $\left(1 - 2\eta \sin \frac{\dot{i}}{2} \,.\, \sin \frac{\dot{j}}{2}\right)$ is to be written for 1. The former correction is sensible for many of the satellites of the solar system, though it does not affect the nature of the results; the latter correction is insensible. The total effect of this correction is a slight increase of the direct as opposed to the cross terms. With these corrections equation (6) should read

$$\left.\begin{aligned} 2gn\frac{di}{dt} &= \tau^2 \left\{ \sin \tfrac{1}{2}(i-j) + \eta \cos \frac{\dot{i}}{2} \sin \frac{\dot{j}}{2} \right\} \sin 4f \\ &\quad - \tfrac{1}{4}\tau\tau_s \left\{ 1 - 2\eta \sin \frac{\dot{i}}{2} \sin \frac{\dot{j}}{2} \right\} \sin 2i \sin 2g + \frac{\tau_s^2}{64} \sin i \\ &\quad \left\{ \begin{aligned} &-12 \sin 4h + \cos i(26 \sin 4f + 12 \sin 2g) \\ &+ \cos 2i \,.\, 12 \sin 4h + \cos 3i(6 \sin 4f - 12 \sin 2g) \end{aligned} \right\} \end{aligned}\right\} \quad 6\,(a)$$

For practical applications it becomes necessary to extend this

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result so as to include the case of several small satellites; that is done in the following part.

IV. *Extension to the Case where several Small Satellites are Present.*

§ 1. The result given in equation (6) may be easily modified so as to give the value for $\frac{di}{dt}$ in the presence of several satellites. The terms arising from case (i) will be obtained by a simple summation $\Sigma_r \tau_r^2 \sin \frac{1}{2}(i-j_r) \sin 4f$, where r is a suffix denoting one of the satellites. If we consider the mutual action of two satellites (r and p) we obtain from cases (ii) and (iii), replacing τ_s by τ_p, $-\tau_r\tau_p \sin \frac{1}{2}(i-j_r) \sin 2g$, and $-\tau_r\tau_p \sin \frac{1}{2}(i-j_p) \sin 2g$. The action of the satellites on the solar tides is $-\frac{1}{4}\Sigma_r \tau_r \tau_s \sin 2i \sin 2g$, obtained by a simple summation from the previous result. As before, the action of the Sun on the satellite's tides is negligible in comparison. The action of the Sun on its own tides is unaffected, and we get eventually, if r, p signify any two of the m satellites of a planetary sub-system,

$$\left.\begin{aligned} 2gn\frac{di}{dt} &= \sum_{r=1}^{r=m} \tau_r^2 \sin \tfrac{1}{2}(i-j_r) \sin 4f - \sum_{r=1}^{r=m} \sum_{\substack{p=1 \\ r \neq p}}^{p=m} \tau_r\tau_p \sin \tfrac{1}{2}(i-j_r) \sin 2g \\ &\quad -\tfrac{1}{4}\sum_{r=1}^{r=m} \tau_r\tau_s \sin 2i \sin 2g + \frac{\tau_s^2}{64} \sin i\{-12 \sin 4h + \cos i(26 \sin 4f \\ &\qquad + 12 \sin 2g) + \cos 2i \,.\, 12 \sin 4h + \cos 3i(6 \sin 4f - 12 \sin 2g)\} \end{aligned}\right\} \ldots \quad (7)$$

Or if we allow for the correction discussed at the end of the previous Part, this equation should read

$$\left.\begin{aligned} 2gn\frac{di}{dt} &= \sum_{r=1}^{r=m} \tau_r^2 \left(\sin \tfrac{1}{2}(i-j_r) + \eta_r \cos \frac{i}{2} \sin \frac{j_r}{2}\right) \sin 4f \\ &\quad - \sum_{r=1}^{r=m} \sum_{\substack{p=1 \\ r \neq p}}^{p=m} \tau_r\tau_p \left(\sin \tfrac{1}{2}(i-j_r) + \eta_r \cos \frac{i}{2} \sin \frac{j_r}{2}\right) \sin 2g \\ &\quad -\tfrac{1}{4}\sum_{r=1}^{r=m} \tau_r\tau_s \sin 2i\left(1 - 2\eta_r \sin \frac{i}{2} \sin \frac{j_r}{2}\right) \sin 2g + \frac{\tau_s^2}{64} \sin i\{-12 \sin 4h \\ &\quad + \cos i(26 \sin 4f + 12 \sin 2g) + \cos 2i \,.\, 12 \sin 4h + \cos 3i\,(6 \sin 4f \\ &\qquad - 12 \sin 2g)\} \end{aligned}\right\} \quad 7\,(a)$$

§ 2. We proceed to apply this result to the different members of the solar system. *Mercury* and *Venus* have no satellites and they have therefore been already dealt with in Part II. The Earth-Moon system, differing as it does essentially from the other planetary sub-systems, has been already dealt with by Sir George Darwin in "Orbits." The satellites of *Mars* are so very small that for them $\frac{\tau}{\tau_s}$ is small. In fact, estimating their masses on the assumption that they are of the same density as the

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satellite of known mass nearest to them in size, *Mimas*, we have the following approximate table :

Satellite.	Mass in Terms of Primary.	$\frac{\tau}{\tau_s}$.
Deimos	·00000001	$\frac{1}{17}$
Phobos	·0000000003	$\frac{1}{11000}$

Thus the cross terms with the Sun are small compared with the purely solar terms, being equivalent to a reduction of the solar term in (7) $\frac{\tau_s^2}{64}$ 12 sin $2g$ sin i cos i to $\frac{\tau_s^2}{64}$ 10 sin $2g$ sin i cos i. The terms arising from the action of the satellites on each other's tides are negligibly small ; for we have in their case an additional factor, sin $(i-j_r)$, which is not large enough to be sensible to observation. *Mars* will move then almost as if unattended by satellites, in the manner indicated in Part II. The position of equilibrium in its case will be with a somewhat smaller obliquity than if no satellites were present.

We proceed to the consideration of the Jovian system. The table corresponding to this system has been compiled from Professor Young's data * and from Tisserand.†

Satellite.	$\frac{\tau}{\tau_s}$.	$i-j$.
Jupiter V.	Very small	Assumed negligible
Io	103	10″
Europa	35	1′ 7″
Ganymede	32	4′ 38″
Callisto	2·9	24′ 12″

while i is taken as 3° 5′.

We may neglect *Jupiter* VI. and VII. They are so small that their $\frac{\tau}{\tau_s}$ is bound to be very small, and they will have no influence on the motion of the planet's axis. We may note that the key to their high inclination does not seem to lie in tidal friction.‡

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* *Manual of Astronomy*, Tables I. and II. pp. 583–5.

† *Mécanique Céleste*, vol. iv. p. 62.

‡ "Professor W. H. Pickering has indeed suggested that his theory does account for the high obliquities of *Jupiter* VI. and VII., the planet having left them behind as it tilted over into its present position. Owing, however, to the fact that secular alterations in the positions of the satellites' orbits are produced far more rapidly than in the planet's obliquity, it is hardly possible that satellites can have been left behind in the way suggested. The fixed plane about which a satellite's orbit oscillates will always lie between the planet's orbit and the planet's equator, and the most that the change in the planet's obliquity could do would be possibly to increase the amplitude of

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Substituting in (7) we obtain for *Jupiter* :

$$\frac{2gn}{\tau_s^2}\frac{di}{dt} = 1\cdot2 \sin 4f - 5\cdot7 \sin 2g - 4\cdot6 \sin 2g + (\cdot02 \sin 4f + \cdot0001 \sin 2g - \cdot00006 \sin 4h) \quad \ldots \quad \ldots \quad (8)$$

The additional terms on the right-hand side of this equation due to the correction allowed for in 7 (a) are certainly less than

$$+\cdot53 \sin 4f - 2\cdot17 \sin 2g.$$

In any case, then, the predominant terms are those arising from the cross-action of the satellites on each other's and on the solar tides, and they cause the planet's equator to sink steadily down into the ecliptic.

In a similar way we derive from data given in Struve's *Memoir* * the following table for *Saturn's* system :

Satellite.	$\frac{\tau}{\tau_s}$.	$i-j$.
Mimas...	9·58	0
Enceladus	15·4	0
Tethys...	35·8	0
Dione	28·8	0
Rhea	22·7	2′·79
Titan	96·8	37′·2

while $i = 26° \; 49'$.

Struve does not give the masses of *Hyperion* and *Iapetus*, but if we use Professor Pickering's photometric measurements † of them and of *Phœbe* as some indication of their diameters, and thence obtain their masses by comparison with satellites of about the same size, we obtain for them the following results :

Satellite.	$\frac{\tau}{\tau_s}$.
Hyperion	·005
Iapetus	·015
Phœbe	·00001

It seems, then, safe to omit these three satellites in considering

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the oscillations about this fixed plane. This does not seem a sufficient explanation for the phenomenon in question, and some other cause, such as the close coincidence in size of the orbits, must apparently be sought to account for it."

* *Publications de l'Observatoire Central Nicolas*, vol. xi., "Beobachtungen der Saturnstrabanten," Hermann Struve.

† *Annals of the Harvard College Observatory*, vol. xi. p. 269.

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changes in the planet's obliquity, and we obtain from (7) for *Saturn*:

$$\frac{2gn}{\tau_s^2}\frac{di}{dt}=51\sin 4f-61\sin 2g-42\sin 2g$$
$$+(\cdot 17\sin 4f+\cdot 03\sin 2g-\cdot 03\sin 4h) \quad \ldots \quad (9)$$

The additional terms due to the correction already noted have as an outside value $+0\cdot 36 \sin 4f - 2\cdot 14 \sin 2g$.

Here again, then, the cross terms prevail and the planet's equator is being slowly driven down into the ecliptic. For *Saturn* and *Jupiter* alike this result seems independent of the viscosity assumed for the planet.

In extending the theory to the systems of *Uranus* and *Neptune* we are handicapped by lack of exact knowledge of the masses of the satellites and of the obliquities of the planets. Using the data as to the satellites' diameters given by Professor Young, and the same assumptions as before as to their densities we obtain the following results:—

Planet.	Satellite.	Mass in Terms of Primary.	$\frac{\tau}{\tau_s}$.
Uranus	Ariel	·0000003	42
	Umbriel	·00000014	8
	Titania	·000009	110
	Oberon...	·0000016	8
Neptune	Lassell's satellite ...	·0004	40,00

A difficulty arises if we try to get some idea of the positions of the equators of these planets, or rather of the secular values of $(i-j_r)$ for their various satellites. For the positions of the equators have not been determined by observations, and the satellites have not yet been observed long enough with sufficient accuracy to enable one to deduce the secular values of their inclinations with any degree of certainty; and naturally the obliquity and oblateness of these planets cannot yet be derived by theoretical means from the motions of their satellites. In view of the great distances of these planets from the Sun it seems tolerably safe to assume a negligibly small value of $(i-j_r)$ for their satellites. In the case of *Uranus* the most likely assumption is that the predominant term is the one arising from the action of the Sun on the satellite's tides; if this is so, then at present the obliquity is increasing from 90° towards 180°. In the case of *Neptune's* satellite we know that η is fairly large, and considering this in conjunction with the large value of $\frac{\tau}{\tau_s}$ we derive from 7 (a) with a considerable amount of certainty the conclusion that the obliquity of *Neptune* is at present also increasing towards 180°.

§ 3. (*a*.) We may now inquire in greater detail what this theory points to as the probable past history of the outer

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planetary system. With regard to *Jupiter* it requires that when the satellites were first evolved solar tidal friction had already reduced the obliquity below 90°. For otherwise *Jupiter's* obliquity, instead of decreasing, would have increased from 90° and would at present be nearly 180°; and it does not seem possible that *Jupiter* VI. and VII. would have obliquities less than 90° at present, if they had been evolved while *Jupiter* had an obliquity greater than 90°. For *Saturn*, on the other hand, it seems necessary that *Phœbe* should have been evolved while the obliquity was still greater than 90°; possible that *Iapetus* and *Hyperion* were also evolved then, and impossible for *Titan* and the inner satellites to have been existing then. In that case the equation $K \sin 2j = K' \sin 2(i-j)$ connecting i and j would have led to the following state of affairs: When *Phœbe* was evolved its orbit would gradually move down to the ecliptic, as for it $\frac{K}{K'}$ is large. In the case of *Iapetus* $\left(\frac{K}{K'}\right.$ near and rather less than unity$\left.\right)$ the orbit would move from the planet's equator down to the ecliptic through an angle certainly less than $(180°-j)$. *Hyperion* would remain near *Saturn's* equator. As *Saturn's* axis slowly moved, under the influence of the solar tides and practically undisturbed by these three satellites, into the plane of *Saturn's* orbit round the Sun, *Phœbe* would fall more and more closely down to the ecliptic, while *Iapetus* would fall back upon the planet's equator. For only thus could the equation $K \sin 2j = K' \sin 2(i-j)$, $(K \neq K')$ be satisfied for $i = 90°$. *Hyperion* and *Iapetus* then would follow *Saturn's* equator over, while *Phœbe* would settle down with an obliquity near 180°. It is necessary that *Titan* and the other satellites should not be evolved with $i > 90°$, for in that case the cross term given by the action of the Sun on their tides would have been the predominant term, and *Saturn* would have moved down to an obliquity of 180° instead of one of 0°.

The case of *Uranus* seems to require that its satellites were evolved while the planet had an obliquity greater than 90° and less than its present value of 97° 51′. For the expression for $\frac{di}{dt}$ consists of two parts. One part, due to the cross terms, varying with $-\sin 2i$ and driving the obliquity away from 90°, is at present apparently the most important. The other, or solar, part tending to diminish the obliquity, decreases in value as i approaches 90°, but not so rapidly as $\sin 2i$ and not to the value zero. Since the first part prevails now and $i > 90°$, the obliquity can never, since the evolution of the satellites, have been near enough to 90° for the solar part to have been predominant, for then the obliquity would have become less than 90° and have remained so. And, since the obliquity is at present increasing, the satellites must have been evolved while the planet had an obliquity greater than 90° and less than its present value.

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Neptune's one satellite too must have been evolved while *Neptune* had an obliquity greater than 90° for it to have ever reached its present position. At present, by its action on its own tides, it appears to be driving *Neptune* towards a position with an obliquity of 180°.

This theory accords well with Sir George Darwin's view * that the outer planets should evolve their satellites at an earlier period in their evolution than the inner ones. But it leaves us with a serious difficulty with regard to time, a difficulty which it is impossible to face properly. For with elements of uncertainty arising from the changing viscosity, density, and rotation period of the planet, with further complications of an unknown nature springing from the birth of a satellite and from an inability to settle at what period in the planet's evolution the satellite's tides will rise into importance, and with the greatest unknown factor remaining in the effects of heterogeneity, any results that might be obtained by integration would have very little value and would form a very inadequate return for the toil and labour involved.

Nevertheless it seems advisable to consider several points closely connected with the time difficulty and more open to examination. For instance, the question arises how far, if the planets tilted over under the influence of tides raised by the Sun, the dimensions of their orbits would be simultaneously altered. From "Evolution" (p. 534) we see that the rotational momentum of any planetary system is so minute compared with its orbital momentum round the Sun, that even the complete tilting over of a planet due to solar tidal friction would not require any appreciable compensating change in its mean distance from the Sun to conserve the angular momentum of the system. A finite change of tilt of the planet's equator is possible with only an extremely small alteration in the planet's mean distance. This is confirmed by a comparison of the value of $\frac{di}{dt}$ with the value of $\frac{1}{c}\frac{dc}{dt}$ derived from "Orbits," equation (70). This shows that for changes of obliquity in the solar system caused by tides set up in the planets we may consider the dimensions of the planets' orbits as fixed.† This conclusion is not affected when we consider the

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* "Evolution," p. 532.

† Omitting a variable factor, a function of i, $\frac{1}{c}\frac{dc}{dt}$ is of the same order as $\frac{di}{dt}$ (expressed in circular measure) multiplied by

$$\left\{\frac{\text{the orbital momentum of the planet}}{\text{the rotational momentum of the Sun}} \times \frac{\text{rate of rotation of the Sun}}{\text{rate of rotation of the planet}} \times \left(\frac{\text{radius of planet}}{\text{radius of Sun}}\right)^2 \times \left(\frac{\text{mass of planet}}{\text{mass of Sun}}\right)^2\right\}.$$

At present for the solar system the greatest value of the above expression is about $\frac{1}{200,000,000}$, and it does not seem likely that it has been much greater for any of the planets in the past.

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similar effects caused by tides in the Sun; for so long as the viscosities are small these tides will only give rise to terms in $\frac{dc}{dt}$ negligible compared with those already considered.

Taking the constancy of the mean axes of the planets as established as far as this investigation is concerned, we will now compare the relative size of the couples which the Sun exercises upon the various planets, or rather of the resulting rates of change of obliquity. Omitting the variable factor, arising from the function of i, which is always present in the expression for $\frac{di}{dt}$, and taking the relative rate to be represented by $\frac{r^2}{gn}\sin 4f$, further making the arbitrary assumption that the viscosity (supposed small) varies with the density, we obtain the following table:—

Planet.	Rate of Change of Obliquity.
Mercury	222,000,000
Venus	1,440,000
Earth	140,000
Mars	77,200
Jupiter	1
Saturn	$\frac{1}{7}$
Uranus	$\frac{1}{130}$
Neptune	$\frac{1}{4200}$

If, then, tidal friction is in the main responsible for the present obliquities of the planets we are forced by the figures of the above table to one of the following assumptions. Either the outer planets were evolved at periods long anterior to the inner ones, and therefore are relatively more advanced, or else they had initially very different obliquities, the outer ones being more removed from the natural obliquity of 180° than the inner one;* a third alternative remains, that the inner planets have spent long periods in or near positions of equilibrium, where they have been moving very slowly or not at all; the result of this would be that they would fall behind the positions which, by comparison with the steadily moving outer planets, they should have reached. Possibly the present state of affairs is due to a combination of these three causes.

There remains one other difficulty in connection with the time required for the working out of the theory, and that difficulty, though an almost necessary accompaniment of any such theory, would be alone sufficient to prevent one from urging its acceptance on dynamical grounds alone. It does not appear

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* This might well be the case in the spiral nebular theory outlined by Professor Moulton (*Astrophysical Journal*, 1905 November).

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that, for such enormous periods of time as we are here concerned with, our ordinary dynamical equations are of sufficient exactitude to prevent the entrance of some unknown factors, which may profoundly modify the course of the evolution of the system. This difficulty must be regarded as an additional cause for receiving the theory with all reserve.

§ 3. (*b*.) We have seen that the mean distances of the planets from the Sun may be regarded as unaffected by tidal friction. It is of interest to discuss how far, as hinted in "Evolution," and so far tacitly assumed in this paper, the same holds good for the orbits of the satellites. A comparison of the value of $\frac{1}{c}\frac{dc}{dt}$ for a satellite[*] and of $\frac{di}{dt}$ for a planet shows at once that so long as the satellite remains tidally effective (*i.e.* so long as τ_r is of the same order as τ_s) the satellite's orbit will increase very largely, while i will only vary by an extremely minute quantity. In fact, while a satellite's mean distance may be enlarged by tidal friction so much that the satellite's tides become negligible compared with the Sun's tides, the planet will not appreciably alter its obliquity.[†] The result of this seems to be that the action of the satellites may be neglected in a discussion of the changes of obliquity of a planet due to tidal friction; the prevailing term, save only for a short interval, is that due to the action of the Sun on its own tides. If this be so, then on examining the present state of affairs in the solar system we are faced by difficulties which had been previously settled by the action of the satellites. The present small obliquity of *Jupiter*, requiring an almost impossibly great viscosity if explained by solar tidal friction alone, had been regarded as a natural consequence of the tidal action of the satellites. And the large angle through which *Saturn* had tilted since the evolution of *Phœbe* had been looked upon as in great part due to the tidal action of its satellites.

The following considerations may perhaps lead to a solution of these difficulties. It will have been noticed that in the previous discussion of the Saturnian system we omitted the effect of the rings on the planet's obliquity. Now the tidal theory of an annular satellite has been briefly sketched in "Precession" (p. 518), and it has been shown that the resultant couple on a planet is practically the same as is obtained from the cross action of a large number of satellites on each other's tides. While all the secular terms in the expression for the rate of change of the dimensions of the ring vanish a secular change is produced in the planet's obliquity by the tides set up by the ring. This we can express as follows: If a, s, r are suffixes referring to the ring, the Sun,

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[*] Derived from "Orbits," (68) and (73).

[†] An exception to this is the Moon; and the true significance of the exceptional nature of the Earth-Moon system pointed out in "Evolution" seems to be that in its case finite changes of obliquity and mean distance accompany one another; the one does not proceed very much more rapidly than the other.

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and any satellite of the planet respectively, then the new terms in $2gn\frac{di}{dt}$ to be added to (7) are

$$\left\{-\tfrac{1}{2}\tau_a{}^2 \sin \tfrac{1}{2}(i-j_a)-\sum_{r=1}^{r=m}\tau_a\tau_r\left(\sin \tfrac{1}{2}(i-j_a)+\sin \tfrac{1}{2}(i-j_r)\right) -\tfrac{1}{4}\tau_a\tau_s \sin 2i\right\} \sin 2g. \quad (10)$$

In the case of *Saturn's* rings, if we accept Struve's figures for $i-j_a$, namely, $0'{\cdot}74$, and Tisserand's avowedly doubtful estimate of their mass as $\frac{1}{1000}$ mass of *Saturn*, we obtain the following approximate value for $\frac{\tau_a}{\tau_s}$:

$$\frac{\tau_a}{\tau_s} = 62700.$$

With this value we obtain from (10) the following terms to be added to the expression for $\frac{2gn}{\tau_s{}^2}\frac{di}{dt}$ for *Saturn* given in (9):

$$-(21600+346000+36300) \sin 2g \text{ or } -404000 \sin 2g.$$

This is, of course, only a very rough approximation, but it is sufficient to show that the terms due to the ring, while of the same sign as those due to the cross action of the satellites, far outweigh them in magnitude. At the same time they are not apparently affected with that transitory nature which renders the satellites' tides ineffective in producing large permanent changes of obliquity in the planet. With the above figures representing the magnitude of the tides set up by the ring the difficulty referred to above disappears for *Saturn*; the further difficulty of the small obliquity of *Jupiter* also ceases to have any force if we assume that *Jupiter's* satellites have in the course of their evolution passed through an annular stage.

We may say, then, that the theory of planetary inversion suggests, but does not absolutely require as a condition for its truth, an annular stage in the history of the satellites of *Jupiter* and *Saturn*. More than this we do not care to state till a more detailed application of the tidal theory has been made to the case of a planet attended by a group of satellites. The very doubtful question whether perturbations in a ring of satellites could ultimately lead to the formation of one or several satellites must also be discussed before the difficulties considered in this section can be removed.

V. *Final Conclusions and Summary of Results.*

The previous part concluded with a mention of one or two difficulties which, unlike the more abstract difficulties connected with time and heterogeneity, may yield to analytical treatment.

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on further investigation. Some explanation may be required for the appearance of this paper is so incomplete a form as is thus implied, and I may perhaps plead in excuse the desire not to overburden this already long paper with a discussion of the details rather than of the principles involved in the theory of planetary inversion. A further reason for not delaying the publication of this paper any longer may be found in the discussion on this subject between Professor W. H. Pickering and Professor F. R. Moulton, which has been going on in the columns of the *Astrophysical Journal.** And I should like to say a few words about their remarks on the subject.

Stimulated though one may be by Professor Pickering's persevering and successful search for a new satellite of *Saturn*, by his bold and equally successful idea of a retrograde orbit for it, and by his suggested explanation for it, one must be careful not to be misled by the vague terms in which he described his theory. These left ample room for the misunderstanding of the theory which, I believe, has invalidated Professor Moulton's criticism of it. The latter's original misconception led him to some further assumptions, which the above analysis of the theory has shown to be unnecessary, but which fundamentally affect the validity of his criticism.

In conclusion I will sum up briefly the view to which the present application of Sir George Darwin's tidal theory has led me. And first let me give a short summary of the results obtained in the investigation. In Part II. by an extension of the results previously obtained by Sir George Darwin I showed that for a planet moving round the Sun unattended by any satellite the stable position was with an obliquity between 0° and 90°, near 90° for all but large viscosities, and steadily decreasing in value as the viscosity increased. Any planet starting with an obliquity greater than 90° would, under the influence of solar tidal friction, tilt over until its obliquity had decreased below that value; this would hold good so long as no satellites with large tidal influence compared with the Sun were evolved.

In Part III. I proved that, if a satellite were present of the type that we find in the solar system, tidal friction would have no sensible effect on the position of its orbit. I further obtained the additional terms, due to the second satellite, in the expression for the secular changes in the planet's obliquity. In Part IV. I extended these results so as to admit of several satellites being present, and obtained a formula leading to the following conclusions: The direct terms arising from the action of a satellite on its own tides tend to drive the planet's axis of rotation into the ecliptic. The cross terms arising from the action of the satellites on each other's tides and on the solar tides tend to drive the axis of rotation out of the ecliptic into a

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* 1905 December.

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position making an angle of 90° with it; these are the predominant terms in most of the planetary sub-systems at present, and thus in the presence of a group of satellites of large tidal influence the stable value for the obliquity will be 180° or 0°, and the planet will move away from an unstable equilibrium position with an obliquity of nearly 90°. One exceptional case exists: if one of the satellites has a tidal influence far in excess of that of any of the others, and if this satellite is not too close to the planet, so that its proper plane does not make too small an angle with the planet's equatorial plane, then the prevailing term may be the one arising from the direct action of this satellite on its own tides, and the equilibrium position will be with the planet's axis of rotation nearly in the ecliptic, or with an obliquity just less than 90°.

In the same part I discussed the application of this theory, if true, to the past and present history of the solar system, and I came to the following conclusion: *Mercury*, *Venus*, and *Mars* have tilted over under the influence of solar tidal friction in accordance with the theory of Part II. Either under the influence of bodily tides of great viscosity they have gained an equilibrium position of small obliquity, or under the influence of ocean tides of small viscosity they are returning to a position of equilibrium with an obliquity of nearly 90°. Observation does not settle which of these two alternatives represents the case of *Mercury* and *Venus*, though for *Mercury* I incline to the former. It seems probable that the latter alternative is required to explain the obliquity of *Mars*. The case of the Earth I have omitted, as already fully dealt with by Sir George Darwin; but it is possible to fit into this theory the initial requirements of his discussion of the evolution of the Earth-Moon system. *Jupiter* must have evolved its satellites after its obliquity had decreased below 90°; partly under their influence it has been driven down towards a stable position of small obliquity, which it has now nearly reached. *Saturn* shed *Phœbe*, and possibly also *Iapetus* and *Hyperion*, while its obliquity was greater than 90°; as under solar tidal influence it passed through the critical position, where its obliquity was 90°, *Phœbe* sank down into the ecliptic in a retrograde orbit, while *Iapetus* and *Hyperion* moved over with the planet's equator. Afterwards the inner satellites were evolved, and under their influence and the influence of the rings *Saturn's* obliquity has steadily diminished—and is still diminishing—towards a small stable value. As seems highly probable for a planet further removed from the Sun, and therefore less likely to have its increasing rotation checked by solar tidal friction, the satellites of *Uranus* were evolved in an earlier stage of its evolution, before its obliquity had decreased to 90°; they have stopped the decrease in obliquity, which would arise from the solar action, and they are now driving *Uranus* back to a stable position with an obliquity of 180°. *Neptune*, with its one satellite of extremely large tidal influence, is being driven towards an equilibrium

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position with an obliquity of 180°. I should add that uncertainty as to the data for the satellites of *Uranus* and *Neptune* leaves even the present direction of motion of their equators very doubtful, but that the results above given seem on the whole the most probable.

In the concluding section of this part I gave reason to believe that the effect of the satellite's tides on the obliquity of the planet might only be transient. It seems very possible, from such investigation as I have made into the tidal theory of a system of satellites, that their orbits would be enlarged by tidal friction very rapidly compared with the rate at which the planet would tilt over; thus they would not remain tidally efficient long enough to perceptibly affect the obliquity of the planet. If this should be the case—and it seems probable—then I suggest as the easiest explanation of certain remaining difficulties that the satellites of *Jupiter* and *Saturn* have passed through an annular form at some previous stage in their history. This latter idea is not essential to the successful working out of the theory; at present it is only put forward very tentatively indeed, and as a subject for further research.

Viewed broadly, then, the theory of planetary inversion, though it entails some difficulties of detail, remains a tenable hypothesis. As explained by Sir George Darwin's tidal theory it involves three main assumptions: (1) that the outer satellites of a planet were evolved before the inner ones; (2) that the determining factor producing secular alterations in a planet's obliquity has been tidal friction; and (3) that the time involved in the scheme is not so great as to invalidate the ordinary dynamical equations. A justification for these assumptions may perhaps lie in the satisfactory explanation which the theory affords both of the large obliquities of *Uranus* and *Neptune* and of the presence of a satellite such as *Phœbe.* The secular motions with which the theory is concerned are so extremely slow that it can hardly yet be proved or disproved by reference to the gravitational theory of the motions of planets and their satellites;* the theory would gain some support by the discovery of satellites to *Uranus* and *Neptune* of the same type as *Phœbe*, if their motion were retrograde; it would be overthrown if their motion were direct. The theory remains then at present a speculative hypothesis, which is on the whole well supported by the theory of tidal friction, and which gives the only explanation so far offered for certain facts.

My indebtedness to Sir George Darwin's previous work on the tides has been apparent throughout this paper, but I wish to take this opportunity of thanking him most heartily for the helpful advice and kind encouragement which he has given me whilst writing it.

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* This was written before I had seen the paper in which Mr. Cowell successfully explained the apparent secular acceleration of the Sun and Moon by an application of the theory of tidal friction.

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